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Dimensional Analysis, Using Computer Symbolic Mathematics*

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This paper describes a computer-algebra program that performs automatic dimensional analysis. The user merely provides a list of the types of physical quantities involved in a problem, and the program produces a minimum-size set of nondimensional groupings of these quantities. These groupings reveal qualitative information about the solution, and they may reduce the necessary number of variables that must be varied independently in an experimental or numerical solution of the problem.

1. INTRODUCTION

Dimensional analysis provides a systematic procedure for combining physical quantities into dimensionless groups, which often reduces the number of variables necessary to describe a physical problem. A reduction in the number of variables reduces the necessary number of experiments or numerical solutions necessary to solve the problem over the full range of its physical variables, and a reduction also reduces the number of tables or graphs necessary to present such solutions. Moreover, the groupings also reveal qualitative relations among the physical variables.

Section 2 summarizes the physical and mathematical foundations of dimensional analysis, and Section 3 shows examples of the use of a program implementing these techniques. A related paper by Stoutemyer [6] shows how computer algebra may also be useful for automatic units conversion and consistency checking when computing with expressions which are not dimensionless.

2. MATHEMATICAL AND PHYSICAL FOUNDATIONS

The units of all known classes of physical quantities may be expressed as simple products of rational powers of the units of a few primary classes of physical quantities. For example, a unit of the class of all accelerations may be expressed as a unit of the class of all lengths divided by the square of a unit of the class of all times.

The primary basis can be selected as the set of classes of all masses, lengths, times,

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temperatures, charges, and forces. The units of all known classes of physical quantities may be expressed in terms of the units of these classes. For example, letting { } denote the units of its argument,

$$\{\text{energy}\} = \{\text{momentum}\}\{\text{velocity}\}, \tag{1}$$

$$\{\text{momentum}\} = \{\text{mass}\}\{\text{velocity}\}, \tag{2}$$

$$\{\text{velocity}\} = \{\text{length}\}/\{\text{time}\}. \tag{3}$$

If we choose to regard Newton's third law as the definition of the unit of force in terms of those of momentum and time, with a dimensionless proportionality constant of 1, then

$$\{\text{force}\} = \{\text{momentum}\}/\{\text{time}\}. \tag{4}$$

If we choose to regard Coulomb's law as defining unit of force in terms of units of charge, with a dimensionless proportionality constant of 1, then

$$\{\text{force}\} = \{\text{charge}\}^2/\{\text{length}\}^2. \tag{5}$$

If we regard Boltzmann's law as a definition of the unit of temperature in terms of the unit of the average energy per degree of freedom per molecule for a perfect gas, with a dimensionless proportionality constant of 1, then

$$\{\text{temperature}\} = \{\text{energy}\}. \tag{6}$$

(The number of molecules and the degrees of freedom are pure numbers.)

If we regard Newton's law of gravitation as defining unit of force in terms of those of mass and length, with a proportionality constant of 1, then

$$\{\text{force}\} = \{\text{mass}\}^2/\{\text{length}\}^2. \tag{7}$$

If we regard Einstein's law of mass–energy equivalence as defining the unit of energy in terms of that of mass, with a proportionality constant of 1, then

$$\{\text{energy}\} = \{\text{mass}\}. \tag{8}$$

If we regard deBroglie's law relating the wavelength and momentum of a photon as a definition of the unit of momentum in terms of that of length, with a nondimensional proportionality constant of 1, then

$$\{\text{momentum}\} = 1/\{\text{length}\}. \tag{9}$$

Together, Eqs. (1) through (9) imply

$$\{\text{force}\} = \{\text{velocity}\} = 1, \tag{10}$$

$$\{\text{length}\} = \{\text{time}\} = \{\text{momentum}\} = \{\text{temperature}\} = \{\text{energy}\} = \{\text{mass}\} = \pm 1, \tag{11}$$

$$\{\text{charge}\} = \pm 1. \tag{12}$$

All physical quantities are dimensionless in the system of units given by (10) through (12). (If the concept of a finite universe is disturbing, how about the concept of a dimensionless universe?) Equations (10) through (12) reveal that the speed of light, Boltzmann's constant, and Planck's constant are all simply consequences of our using a dimensional basis that is redundant in view of known physical laws. Different philosophers have said that all is motion, all is matter, all is number, and all is mind. Perhaps all are right.

Could the independent alternatives for the sign in (12) be connected with the fact that both positive and negative charges are observed, and could the independent alternatives for the sign in (11) be connected with antimatter?

At the present time it is fashionable to take the mass, length, and time as the basis, retaining the laws of gravitation, mass–energy equivalence, and photon momentum–wavelength equivalence as side conditions that all physical phenomena are constrained to obey. However, in celestial mechanics, (7) is sometimes used to eliminate mass from the basis, whereas in high-energy physics, (8) and (9) are sometimes used to eliminate all but length from the basis. In general we may regard a set of relations such as a subset of (1) through (9) as a set of simultaneous nonlinear equations to be solved for some of the variables in terms of a remaining basis. By tradition, (7) through (9) are usually excluded, and the relations are solved in terms of mass, length, and time, sometimes together with charge if (5) is excluded and/or temperature if (6) is excluded. However, there are numerous other possible bases.

In terms of mass, length, and time, (7) through (9) are, respectively, equivalent to

$$\{\text{mass}\} = \{\text{length}\}^3/\{\text{time}\}^2, \tag{13}$$

$$\{\text{length}\} = \{\text{time}\}, \tag{14}$$

$$\{\text{time}\} = \{\text{mass}\}\{\text{length}\}^2. \tag{15}$$

One or more of these three equations can be solved simultaneously in a variety of ways to eliminate a corresponding number of basis elements.

All known physical laws are expressible in a dimensionless form that is independent of units. In order to comply with this *invariance* principle, if we use a basis other than (10) through (12), the physical quantities that enter a physical relation cannot necessarily enter it in an arbitrary manner. Rather, the quantities must enter the relation in a dimensionally homogeneous manner so that the relation can be converted to a dimensionless form. There is one such constraint for each class of physical quantities that we include in our basis; and each of these implied constraints might reduce the number of variables in the dimensionless form by 1, compared to the number of physical variables. If we had begun with a basis of fewer classes and sufficient insight, we might have used correspondingly fewer physical variables from the beginning. Fortunately, there is an algorithm that automatically constructs an appropriate set of dimensionless quantities from any given physical quantities and any given basis.

Suppose that we propose the existence of a physical relation relating physical quantities $q_1, q_2, \ldots, q_n$:

$$g(q_1, q_2, \ldots, q_n) = 0. \tag{16}$$

Suppose also that for $i = 1, 2,..., m$, a unit u_i of q_i is expressible in terms of the units b_j of a chosen basis set of primary classes of physical quantities

$$u_i = f_i(b_1, b_2, ..., b_m), \tag{17}$$

and that

$$f_i(\tau_1 b_1, \tau_2 b_2, ..., \tau_m b_m) = \tau_1^{a_{i1}} \tau_2^{a_{i2}} \cdots \tau_m^{a_{im}} f_i(b_1, b_2, ..., b_m). \tag{18}$$

Here τ_j is an arbitrary pure-number scaling factor, and a_{ij} is a pure-number rational exponent, with $i = 1, 2,..., n$ and $j = 1, 2,..., m$.

Let r denote the rank of the matrix with elements a_{ij}, and without loss of generality, order q_i and b_j so that

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{S} \\ \mathbf{C} & \mathbf{T} \end{bmatrix}, \tag{19}$$

with $\mathbf{B}$ r-by-r nonsingular. Also, let

$$v_k = q_k, \qquad k = 1, 2,..., r, \tag{20}$$

$$w_l = q_{r+l}, \qquad l = 1, 2,..., n - r, \tag{21}$$

$$\mathbf{P} = \mathbf{CB}^{-1}. \tag{22}$$

THE BUCKINGHAM PI THEOREM. *Given* (16) *through* (22), (16) *must be expressible in the form*

$$h(\gamma_1, \gamma_2, ..., \gamma_{n-r}) = 0, \tag{23}$$

where γ_l *is the dimensionless product given by*

$$\gamma_l = w_l \prod_{k=1}^{r} v_k^{p_{lk}}, \qquad l = 1, 2,..., n - r. \tag{24}$$

Proof. Brand [1] or Kurth [2, 3].

Note that Eq. (24) implies that q_k is not involved if column k of $\mathbf{P}$ is zero, that there is no relation among any of q_1 through q_n alone if $r = n$, and that q_1 through q_n are all nondimensional if $r = 0$. Note also that $\mathbf{A}$ here is the transpose of the $\mathbf{A}$ in most statements of the theorem.

Compared with (16), (23) has the advantage that there are r fewer variables, simplifying the acquisition and presentation of an experimentally or numerically obtained relation. Equation (23) also gives valuable qualitative information about the effects of scale changes in experimental or numerical models. It is important to note, however, that the nondimensional products are nonunique. Different orderings of $\mathbf{q}$ and $\mathbf{b}$ consistent with (19), will give different γ, and any γ may be mapped into another $\hat{\gamma}$, where each component of $\hat{\gamma}$ is a distinct product of powers of the components of γ. Note that in (23) each component of $\mathbf{w}$ occurs in only one γ_l. Therefore, subject to (19), it is desirable to order the components of $\mathbf{q}$ so that $\mathbf{w}$ contains physical quantities which we would most like to have isolated in a single component of γ. Usually, we would prefer to isolate whatever physical quantity we regard as dependent and also the physical quantities that are most readily varied experimentally or numerically. It is usually less important that physical quantities such as physical constants occur in only one γ component.

3. The Program and Examples of Its Use

Informally, (16) through (24) indicate a direct sequence of steps for constructing γ, from **q**, **b**, and **A**. The corresponding MACSYMA program is available from the author. As a highly important manner of user convenience, the program also constructs **A** from **q** and **b** together with a rather thorough library of over 50 predefined relations such as (1) through (4). There are also mechanisms for the user to add or delete library entries. Judging from my attempts, manual derivation and entry of **A** is alarmingly error prone. (Judging from the percentage of errors that the program has revealed in published examples, manual derivation of γ from **A** and **q** is also rather error prone.) To further minimize the chance of errors in entering **q** and new library entries, they are entered symbolically rather than as vectors of numbers. Similarly, the components of γ are displayed symbolically to minimize the chance of misinterpretation. Steps corresponding to (19) through (22) and (24) could be programmed in a traditional numerical computation language such as FORTRAN, but a minimal necessity for input and manual preprocessing together with a natural style of input and output dramatically help make a program a joy rather than a burden to use.

Although MACSYMA does have a built-in function for symbolically solving simultaneous nonlinear polynomial equations, it is currently impractical to solve a system of over 50 such equations. Consequently, rather than offer the user a completely general choice of basis, he is offered a choice of the set [charge, temperature, length, time, mass] or any subset of this set which excludes charge and temperature if it excludes any of the other three. This selection includes all of the usual bases, while permitting the following technique for avoiding the necessity of solving numerous simultaneous nonlinear equations.

A set of over 50 secondary quantities are predefined in terms of charge, temperature, length, time, and mass by equations such as (1) through (4), with the secondary quantity on the left side. If the user establishes the electric permittivity of a vacuum as a pure number, then ($\text{mass}^{1/2}\ \text{length}^{3/2}/\text{time}$) is substituted for charge in the right side for any secondary quantity that is used. If the user establishes Boltzmann's constant as a pure number, then ($\text{mass length}^2/\text{time}^2$) is substituted for temperature in the right side of any secondary quantity that is used. Next, the right side of (13), (14), or (15) is similarly substituted for the left if, respectively, only one of the gravity constant, speed of light, or Plank's constant is established as a pure number by the user. There are, of course, other possibilities for which basis-class is eliminated by which equation, but it is the constants taken as pure numbers rather than the classes in the basis that are relevant for dimensional analysis. If the user establishes two of these three constants as pure, then a solution of the corresponding two simultaneous equations for the two variables on the left side is substituted in the right side of any secondary quantity that is used. Since there are only three pairs, these solutions are preestablished rather than generated as needed. If the user establishes all three of these constants as pure, then trivially, all physical constants are already dimensionless. The default is to have Boltzmann's constant and the electric permittivity of a vacuum be pure numbers.

MACSYMA is an interactive language, which appreciably enhances the convenience of a program such as this. The language is most fully described by the Mathlab group [1975], but for this demonstration, it suffices to know that for each interaction cycle the user is prompted with a uniquely numbered label beginning with the letter C. The user then types an expression terminated by a semicolon or a dollar sign. MACSYMA then generates a simplified version of the expression having a correspondingly numbered label beginning with the letter D. This result is displayed only if the terminator was a semicolon rather than a dollar sign, but any result may be used by inserting its D-label in a subsequent expression.

Here are a few examples illustrating how the program can be used.

Langhaar [4, pp. 137–139] reports the following example contributed by Knute Takle: It is conjectured that for thermistors there is a physical relationship between the voltage drop, current, ambient temperature, room-temperature resistance, convective heat transfer coefficient, and a constant called β, having the dimension of temperature. First, to see if the dimension of β is already known:

```
(C3) GET(BETA, 'DIMENSION);

(D3)                                   FALSE
```

It is not. To establish it:

```
(C4) DIMENSION(BETA=TEMPERATURE);

(D4)                                   DONE
```

To automatically determine a set of dimensionless variables sufficient to characterize the physical relation:

```
(C5) NONDIMENSIONALIZE([VOLTAGE, CURRENT, TEMPERATURE,
     RESISTANCE, HEATTRANSFERCOEFFICIENT, BETA]);

(D5) [                        VOLTAGE
     [------------------------------------------------------------------,
     [SQRT(HEATTRANSFERCOEFFICIENT) SQRT(RESISTANCE) SQRT(BETA)

              CURRENT SQRT(RESISTANCE)
     ------------------------------------------------,
     SQRT(HEATTRANSFERCOEFFICIENT) SQRT(BETA)

                                          TEMPERATURE]
                                          -----------]
                                              BETA   ]
```

We learn that the relation may be expressed as a function of only the above three variables rather than a function of the six physical quantities. Evidently dimensions were preestablished for all but the last of these particular input quantities, but an appropriate error message would have informed us if this were not so.

As another example in Langhaar [4, pp. 40–41], there is thought to be a relation between the viscosity, average velocity, molecular mass, and repulsion coefficient of a gas. The repulsive force between two molecules is believed to be of the form $K/\text{DISTANCE}^N$ with unknown N, so K must have the dimensions

(C6) DIMENSION(K=MASS*LENGTH↑(N+1)/TIME↑2) $

WARNING: N NOT MEMBER OF [MASS, LENGTH, TIME, CHARGE, TEMPERATURE].

In order to have the computation time in milliseconds printed automatically:

(C7) CPUTIME: TRUE $

TIME=1 MSEC.

To do a dimensional analysis of the gas–viscosity problem:

(C8) NONDIMENSIONALIZE([VISCOSITY, K, MASS, VELOCITY]);

TIME=526 MSEC.

(D8)
$$\left[\frac{K^{\frac{2}{N-1}}\ \text{VISCOSITY}}{\text{MASS}^{\frac{N+1}{N-1}}\ \text{VELOCITY}^{\frac{N+3}{N-1}}}\right]$$

The physical relation must be expressible as a function of this one dimensionless variable, or equivalently, this variable must equal a constant. Consequently, physical measurements can be used to determine N. It turns out to be in the range 7 to 12 for common gases. For this dimensional analysis problem, the exponent matrix **A** has a symbolic entry $N + 1$, so the step corresponding to (22) could not have been performed by conventional numerical matrix routines.

As a final example by Kurth [3, pp. 3–7], suppose that we conjecture a relation between the deflection angle of a light ray, the mass of a point mass, the speed of light, and the distance from the mass to the point of closest approach

(C9) NONDIMENSIONALIZE([ANGLE, MASS, LENGTH, SPEEDOFLIGHT]);

TIME=416 MSEC.

(D9) [ANGLE]

We learn that there cannot be dimensionless relation connecting all of these quantities and no others. Let us also try including the constant that enters the inverse-square law of gravitation:

(C10) NONDIMENSIONALIZE([ANGLE, MASS, LENGTH, SPEEDOFLIGHT, GRAVITYCONSTANT]);

TIME=514 MSEC.

(D10) $$\left[\text{ANGLE}, \frac{\text{GRAVITYCONSTANT MASS}}{\text{LENGTH SPEEDOFLIGHT}^2}\right]$$

Alternatively, for astrophysics problems such as this, we may prefer to use a dimensional basis in which the gravity constant is taken as a pure number, eliminating one member from our dimensional basis. To append the gravity constant to the default list of pure constants

(C11) %PURE: CONS(GRAVITYCONSTANT, %PURE);

TIME=1 MSEC.

(D11) [GRAVITYCONSTANT, BOLTZMANNSCONSTANT, ELECTRICPERMITTIVITYOFAVACUUM]

To proceed with our analysis:

(C12) NONDIMENSIONALIZE([ANGLE, MASS, LENGTH, SPEEDOFLIGHT]);

TIME=961 MSEC.

(D12) $$\left[\text{ANGLE}, \frac{\text{MASS}}{\text{LENGTH SPEEDOFLIGHT}^2}\right]$$

These examples should indicate the nature of the program and its utility. Many of the examples in Langhaar [4] have been tried, and none required more than two seconds of computation time. However, these times could be appreciably reduced by writing a procedure that combined rank-determination with partitioning and inversion, and by translating the program into LISP or compiling it, rather than merely interpreting it.

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